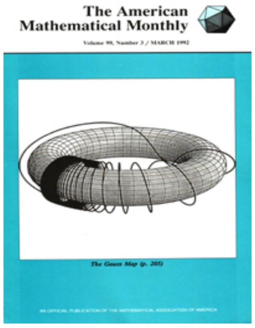

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Continued Fractions and Chaos

R. M. Corless

1. INTRODUCTION. This paper is meant for the reader who knows something about continued fractions, and wishes to know more about the theory of chaotic dynamical systems;[1] it is also useful for the person who knows something about chaotic dynamical systems but wishes to see clearly what the effects of numerical simulation of such a system are. This paper is not purely introductory, however: there are new dynamical systems results presented here and also in the companion paper (Corless, Frank & Monroe [1989]), which presents some discussion of dynamical reconstruction techniques and dimension estimates.

The theory of continued fractions goes back at least to *c.* A.D. 500 to the work of Āryabhata, and possibly as far back as *c.* 300 B.C. to Euclid. The theory of chaotic dynamical systems is relatively recent, going back only to the work of Poincaré [1899] and Birkhoff [1932]. The foundations of the theory of continued fractions, as we know it now, are well established due to the work of Euler, Lagrange, Gauss, and others, while the foundations of chaotic dynamical systems are still evolving. This paper will use the well-established theory of simple continued fractions to explore some current results of the theory of chaotic dynamical systems.

Olds [1963] gives a good introduction to the classical theory of simple continued fractions, by which we mean continued fractions of the form

$$n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

where the n_i are all positive integers, except n_0 which may be zero or negative. We will denote this as $n_0 + [n_1, n_2, n_3, \dots]$, and in what follows n_0 will usually be zero. Simple continued fractions have found applications in Fabry-Perot interferometry (Ikeda & Mizuno [1984]), and the concept of "noble" numbers used in orbital stability and quasi-amorphous states of matter (Schroeder, [1984]). For other uses of simple continued fractions in chaos, see Devaney [1985]. Other types of continued fraction exist, for example, Gautschi [1970], Henrici [1977], Jones and Thron [1980], and others, use functional or analytic continued fractions in approximation theory, since analytic continued fractions can be very effective for computation. We will not be concerned with such continued fractions. We will summarize in the next section all the classical results that we need, without proof. Proofs can be found in Olds [1963], Hardy and Wright [1979], Niven [1956], Khinchin [1963], Billingsley [1963], and Mañé [1987].

[1]One referee has remarked that "This describes the referee, who admits to having found the paper interesting. Though, I suspect, now, more people know about chaos than continued fractions." The author is inclined to agree, and hopes that this paper will interest some of these people in continued fractions.

2. SUMMARY OF CLASSICAL RESULTS

The Gauss Map. We begin with the classical method for finding the continued fraction representation of a number γ. We put n_0 equal to the integer part of γ, by which we mean the greatest integer less than or equal to γ. If the fractional part of γ is not zero, we put γ_0 equal to the fractional part of γ. We then invert γ_0, and put n_1 equal to the integer part of $1/\gamma_0$. Similarly we put γ_1 equal to the fractional part, and repeat. Note that n_0 may be positive, negative, or zero, but that all the subsequent n_i will be positive, and that each γ_i is in the interval $[0, 1)$. This process gives us unique continued fraction for each starting point γ, and the process terminates if and only if γ is rational. (For any rational γ there is one other simple continued fraction which is only trivially different from the one generated by this algorithm.) This algorithm is related to the Euclidean algorithm for finding the greatest common divisor (gcd) of two integers k and m (Olds [1963]), in that if we use this method to find the continued fraction of k/m, then the integer parts that arise are precisely the quotients that arise in the Euclidean algorithm, and in fact the last nonzero remainder from the Euclidean algorithm appears as the numerator of the last nonzero fractional part. This remainder is of course the gcd of k and m. Further, this algorithm can easily be seen to terminate in $O(\log(\min(k, m)))$ operations. Classically, most attention has been paid to the integers generated by this algorithm, which make up the continued fraction itself. However, Gauss was apparently the first to study the other part of this algorithm, which we present as the following map, called the Gauss map (Mañé [1987]) (see FIGURE 1):

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} \bmod 1 & \text{otherwise.} \end{cases}$$

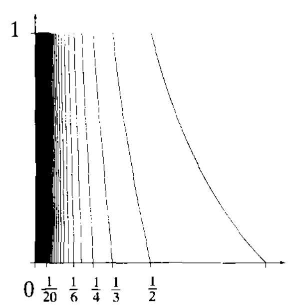

Figure 1. The graph of the Gauss Map $G(x)$. Note that there are an infinite number of jump discontinuities at values of $x = 1/n$, for integers n. In addition, there is a pole at the origin. The darkening of the curve towards the origin is suggestive of the fractional nature of the capacity dimension.

We use the notation "mod 1" to mean taking the fractional part. In terms of the Gauss map G, our algorithm then becomes

$$\gamma_{k+1} = \text{fractional part of } 1/\gamma_k = G(\gamma_k)$$
$$n_{k+1} = \text{integer part of } 1/\gamma_k, \quad \text{for } k = 0, 1, 2, 3, \ldots$$

and we see that the continued fraction is generated as a byproduct of the iteration of the Gauss map. Thus we expect that any classical results on continued fractions will have implications for the dynamics of the Gauss map.

Note that the jump discontinuities occuring at $x = 1/n$ (for each integer n) may all be removed by mapping onto the circle with the transformation $e^{2\pi i x}$. After this is done, we see that the Gauss map ($e^{2\pi i x} \to e^{2\pi i / x}$) is a map of the circle onto the circle, and may be pictured on a torus, as in FIGURE 2. The singularity at the origin is not removed by this transformation. For convenience, the singularity is dealt with by artificially making zero a fixed point of the map (this makes our difficulties no worse). Most theorems on the dynamics of discrete maps assume continuity, which is thus violated here.

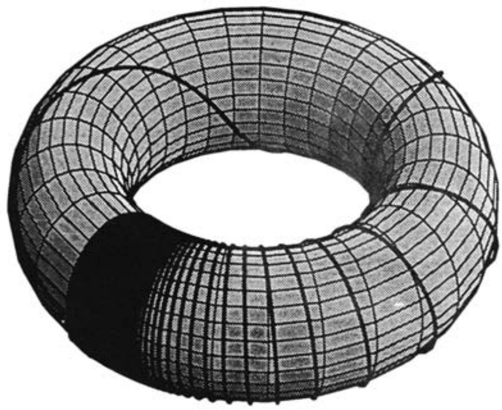

Figure 2. The graph of the Gauss Map $G(x)$ on the torus. Note that all the jump discontinuities have been removed, but that the pole at the origin remains. The darkening of the curve towards the singularity again gives an idea of the fractional nature of the capacity dimension.

We make the following observation: if we represent a point in the interval $[0, 1)$ by its continued fraction, $\gamma_0 = [n_1, n_2, n_3, \dots]$, then a simple induction shows that $G(\gamma_0) = \gamma_1 = [n_2, n_3, n_4, \dots]$, $G(\gamma_1) = \gamma_2 = [n_3, n_4, n_5, \dots]$, $G(\gamma_2) = \gamma_3 = [n_4, n_5, n_6, \dots]$, and so on. This makes a connection between the Gauss map and the "shift map" of symbolic dynamics (Devaney, 1985). We will not explore this connection further here, but we note that the shift maps normally studied are slightly different than the Gauss map, in that here the size of the numbers in the list being "shifted" is not bounded.

An analogy is illuminating: if we think of our space as a circular hoop with the origin at one point O on the hoop, our initial point as a dimensionless bead on the hoop, and the Gauss map is taking the bead from its current position clockwise past O at least once to its next position on the hoop, then the integers n_i are the number of times the bead passes O on the ith iteration (in general the maximum such number is called the "winding number" of the map, and here this is obviously infinite), and the γ_i are the coordinates of the bead on the hoop once it comes to rest. If the bead comes to rest close to the origin on one side, with a small γ_i, then on the next iteration it will be pushed many times around the hoop. If it comes to rest close to the origin on the other side, with a γ_i close to 1, then it will only go past the origin once on its next iteration. We may think of the bead as being pushed around the circle, with the strength of the push being inversely proportional to the distance measured counterclockwise from the point O.

3. DYNAMICAL SYSTEMS TERMINOLOGY. In what follows we give a compact introduction to the terminology used in the study of discrete dynamical systems.

For more details, see Devaney [1985]. To begin with, a **discrete dynamical system** is a recurrence relation $x_{k+1} = G(x_k)$, with the index k playing the role of a discrete "time". Note that the points x_k may be multi-dimensional. The sequence $\{x_k\}_{k=0}^{\infty}$ is called the **orbit** of the initial point x_0 under the **map** $x \to G(x)$, and is denoted as $\text{orb}(x_0)$. Any points x that satisfy $x = G(x)$ are called **fixed points** of the map, and more generally if $x = G^n(x)$ where $G^n(x) = G(G^{n-1}(x))$ then x is called a **periodic point** of the map. If N is the least such number n, then as usual we say x has period N. The **α-limit set** of $\text{orb}(x_0)$ is the set of all initial points whose orbits approach $\text{orb}(x_0)$ as "time" increases; to be precise, an initial point y_0 is in the α-limit set of $\text{orb}(x_0)$ if there exist m and n such that for all $\varepsilon > 0$ there exists K such that $k \geqslant K$ implies $|x_{m+k} - y_{n+k}| < \varepsilon$. The **$\omega$-limit set** of $\text{orb}(x_0)$ is the set of accumulation points of $\text{orb}(x_0)$. An **attractor** of a map is a set of points which "attracts" orbits, from some set of initial points of nonzero probability of being selected. To be precise, an attractor of a map is an **indecomposable** closed **invariant** set Λ with the property that, given $\varepsilon > 0$, there is a set U of positive Lebesgue measure in the ε-neighbourhood of Λ such that if x is in U then the ω-limit set of $\text{orb}(x)$ is contained in Λ and the orbit of x is contained in U (Guckenheimer & Holmes, [1983]). An invariant set is a set such that $G(\Lambda) = \Lambda$, and an indecomposable set is one which cannot be broken into two or more pieces which are distinct under G. A map G is said to be **sensitive to initial conditions** (SIC) if initially close initial points have orbits that separate at an exponential rate. A map that is SIC is also said to be **chaotic**. The possible average exponents of these rates of separation are called the **Lyapunov exponents** of the map. Osledec's theorem (Osledec, [1968]) states that for a wide class of maps, and for almost all initial points, there are only finitely many possible Lyapunov exponents (in fact, only n for an n-dimensional map).

4. CLASSICAL RESULTS INTERPRETED IN DYNAMICAL SYSTEMS TERMINOLOGY PERIODIC AND FIXED POINTS OF THE GAUSS MAP. The following classical theorem, interpreted in a modern dynamical sense, identifies the fixed and periodic points of the Gauss map.

Theorem (Galois). *The number γ has a purely periodic continued fraction, including the first integer n_0, if and only if γ is a "reduced quadratic irrational", which means that γ is a root of a quadratic equation with integer coefficients and, further, that its algebraic conjugate (i.e. the other root of the quadratic) lies in the interval $(-1, 0)$.*

Corollary. *The periodic points of the Gauss map are the reciprocals of the reduced quadratic irrationals.*

For a proof of the theorem, see Olds [1963], or Hardy and Wright [1979]. To prove the corollary, we note that $\gamma = [n_1, n_2, n_3, \dots]$ is periodic under the Gauss map if and only if its continued fraction is periodic, starting at n_1, by the shift property mentioned in the previous section.

An example of particular interest is τ, the golden ratio, which satisfies $\tau^2 - \tau - 1 = 0$. The other root of this quadratic is $-1/\tau$ which is in the desired interval. The continued fraction of τ is $\tau = 1 + [1, 1, 1, 1, \dots]$, so $1/\tau$ has the continued fraction $[1, 1, 1, 1, \dots]$, which shows that $1/\tau$ is a point of period 1 of the Gauss map. We will return to this example later.

There are general results in the theory of chaotic dynamical systems, with which we could hope to establish the character of the set of periodic points of the Gauss map (Šaarkovskii [1964], Štefan [1977], Li and Yorke [1975]). However, these results deal with the characterisation of the behaviour of *continuous* maps of the interval, extended by Block to maps of the circle (Block [1980]), and the Gauss map has a singularity at the origin. Thus the hypotheses of these theorems are not weak enough to apply. However, the results of these theorems hold, as will be seen by direct methods.

We note here that there are infinitely many points of each period. For example, $[n_1, n_2, \dots, n_k, n_1, n_2, \dots, n_k, \dots]$ has period k, for any choice of integers $n_1, n_2, \dots, n_k$. Having points of arbitrary period is one characteristic of a chaotic map (Li and Yorke [1975]). However, we would like to see if the map is sensitive to initial conditions (SIC) in that nearby initial points have orbits that separated at an exponential rate. This again can be established in an elementary fashion by using a classical result.

Theorem (Lagrange). *γ has an ultimately periodic continued fraction, which means that $\gamma = [a_1, a_2, a_3, \dots, a_i, n_1, n_2, \dots, n_k, n_1, n_2, \dots, n_k, \dots]$ with transients $a_1, a_2, a_3, \dots, a_i$ at the start of a periodic continued fraction, if and only if γ is a quadratic irrational (γ is a root of a quadratic with integer coefficients).*

Corollary. *The Gauss map is S.I.C.*

For a proof of Lagrange's theorem, see Hardy and Wright [1979]. To prove the corollary, we note that every rational initial point is "attracted" to the artificial fixed point at 0, while every quadratic irrational is ultimately "attracted" to a periodic orbit. Both sets are dense in the interval $[0, 1)$. The rate of separation may be checked by considering all points in a small interval I, of width ε. By the pigeonhole principle, this interval must contain a rational number of the form p/n, where n is the smallest integer larger than $1/\varepsilon$. The number of iterations of the Gauss map required to reach zero for this initial point is, by the speed of the Euclidean algorithm, $O(\log(n))$, and thus $O(\log(\varepsilon))$. To construct a specific initial point in this interval that does something different under G, first expand p/n into its finite continued fraction: $p/n = [a_1, a_2, a_3, \dots, a_i]$. Then for large enough N, the following infinite continued fraction is the continued fraction expansion of a point in I: $[a_1, a_2, a_3, \dots, a_i, N, 1, 1, 1, 1, \dots]$. Clearly, the orbit of G starting at this initial point winds up on the fixed point at $1/\tau$. Q.E.D.

Aperiodic Points. Of course, non-quadratic irrationals have continued fraction expansions, too. By Lagrange's theorem, these continued fractions are **aperiodic**, and hence the orbit of these initial points under the Gauss map is aperiodic. Note that most numbers in $[0, 1)$ are thus aperiodic. We examine some beautiful examples, beginning with one due to Euler:

1. e (the base of the natural logarithms) has an aperiodic continued fraction expansion $e = 2 + [1, 2, 1, 1, 4, 1, 1, 6, \dots]$. The elements of the orbit of this initial point are always of the form $[1, 2N, 1, 1, \dots]$, $[2N, 1, 1, \dots]$, or $[1, 1, 2N, \dots]$, which tend to 1, 0, and 1/2, respectively. Thus the ω-limit set of this orbit is the set $\{1, 0, 1/2\}$, which, unlike the ω-limit sets of continuous maps, is *not* invariant under the Gauss map since $G(1) = G(1/2) = 0$, so G applied to this set simply gives 0. In other words, we have an asymptotically periodic orbit which is not asymptotic to a real orbit of the map. This cannot happen for a discrete dynamical system with a continuous map.

2. (Stark [1971]). If x is the positive root of $x^3 - 3600x^2 + 120x - 2 = 0$, then

$$x = 3599 + [1, 28, 1, 7198, 1, 29, 388787400, 23, 1, 8998, 1, 13, 1, 10284, 1, 2, 35400776804, 1, 1, \dots]$$

which has very large entries placed irregularly throughout. This intermittency is a typical feature of a chaotic system (Guckenheimer and Holmes [1983]).

3. (Lambert, 1770—*cf* Olds [1963]). The continued fraction for π is not known, in the sense that no pattern has been identified. It begins $\pi = 3 + [7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \dots]$ and some 17,000,000 elements of this continued fraction have been computed by Gosper (Borwein and Borwein, [1987]). There are many open questions about this continued fraction—for example, it is not known if the elements of the continued fraction are bounded.

Lyapunov exponents. We showed earlier that the separation of orbits initially close to each other occurred at an exponential rate. We would like to examine the Lyapunov exponents of the Gauss map, to see if we can explicitly measure the *rate* of separation. The Lyapunov exponents of orbits of the Gauss map are defined as (Devaney [1985])

$$\lambda(\gamma) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\prod_{i=0}^{n} |G'(\gamma_i)|\right)$$

whenever this limit exists. Nearby orbits will separate from the orbit of γ at an average rate of $e^{\lambda k}$, after k iterations of G. Khinchin [1963] derived a remarkable theorem with which we could show the Lyapunov exponent of almost all (in the sense of Lebesgue measure) orbits can be shown to be $\pi^2/6\ln 2$. Easier ways have since been found to establish this result, using ergodic theory. We summarize the ergodic results in the next section. In this section we simply note that for any rational initial point, the above limit does *not* exist. Further, for any periodic orbit the calculation can be made explicitly, to give Lyapunov exponents that *differ* from the almost-everywhere value. For example, the fixed points $\alpha_N = [N, N, N, N, \dots]$ have Lyapunov exponents

$$\lambda(\alpha_N) = 2\ln(1/\alpha_N) \sim \ln(N) + N^{-2} - \tfrac{3}{2}N^{-4} + O(N^{-6})$$

so that there are orbits with *arbitrarily large* Lyapunov exponents, i.e., orbits that are arbitrarily sensitive to perturbations in the initial point. Note also that for the orbit of e, the limit defining the Lyapunov exponent is *infinite*. The special case $N = 1$ gives τ, the golden ratio. Thus $\lambda(1/\tau) = 2\ln\tau = 0.96\dots$, which is smaller than the almost-everywhere Lyapunov exponent. In fact, we have the following:

Theorem. *No orbit of the Gauss map has a Lyapunov exponent smaller than* $\lambda(1/\tau) = 2\ln\tau$.

Proof: Let $\gamma = [n_1, n_2, n_3, \dots]$ be any initial point in $(0, 1)$ such that $\lambda(\gamma)$ exists. We will show that the product $\prod_{i=0}^{N}(1/\gamma_i^2)$ which appears in the definition of $\lambda(\gamma)$ must be at least τ^{2N} (for N sufficiently large) which will prove the theorem. We consider two subsequent elements γ_k and γ_{k+1} of the orbit of γ. If $k = N$, enlarge the product by one term. Note γ_k and γ_{k+1} are related by $\gamma_k = 1/(n_{k+1} + \gamma_{k+1})$. If $\gamma_k \leqslant 1/\tau$ then the contribution of γ_k^{-2} to the product is at least τ^2. If instead $\gamma_k > 1/\tau$ then $\gamma_k \cdot \gamma_{k+1} = \gamma_{k+1}/(n_{k+1} + \gamma_{k+1}) = 1 - n_{k+1}\gamma_k \leqslant 1 - \gamma_k < 1 - 1/\tau = 1/\tau^2$ so the contribution of $1/\gamma_k^2\gamma_{k+1}^2$ to the product is at least τ^4. This proves the theorem.

Remark. There are infinitely many initial points γ in $(0,1)$ with this Lyapunov exponent. For example, all the numbers $\gamma = [n_1, n_2, n_3, \dots, n_k, 1, 1, 1, 1, \dots]$, that is, all the numbers whose continued fractions ultimately end in 1's, have Lyapunov exponent $2\ln\tau$. These are the so-called noble numbers (Schroeder [1984]), noticed for their resistance to chaos, and we see here that they all share the (still positive) minimum possible Lyapunov exponent under the Gauss map.

Ergodic results. The Gauss map is well-known in ergodic theory (see Billingsley [1963] or Mañé [1987]). The results are summarized here, for contrast with the results of the sections previous and following. This section is meant more as incentive for the reader to investigate ergodic theory than as exposition. The Gauss map preserves the Gauss measure

$$\mu(A) = \frac{1}{\ln 2}\int_A \frac{1}{1+x}\,d\lambda,$$

where λ is the Lebesgue measure. Thus the Gauss map is ergodic, and almost all (in the sense of either the Lebesgue or Gauss measure) initial points have orbits which have the interval $[0,1]$ as ω-limit set. Thus the *only* attractor whose basin of attraction has nonzero measure is the interval $[0,1]$. By the ergodicity of the map, we may explicitly calculate the Lyapunov exponent as follows:

$$\lambda(\gamma) = -2\lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=0}^{n}\ln(\gamma_i)\right) = \frac{-2}{\ln 2}\int_0^1\frac{\ln(x)}{1+x}\,d\lambda = \frac{\pi^2}{6\ln 2} = 2.3731\dots,$$

which holds for almost all initial points γ. This is of interest, since there are few nontrivial maps for which the Lyapunov exponent can be calculated explicitly.

5. THE FLOATING-POINT GAUSS MAP. All of the results of the previous sections are valid for the familiar domain of the real numbers. However, when we work in any fixed-precision system, we have two difficulties:

1. Not all real numbers are even representable in the system, and
2. Arithmetic doesn't have the properties we are used to.

For example, defining $\mathbf{u}$ as the smallest machine representable number which when added to 1 gives a number different from 1 when stored, we see that $G(\delta)$ is computed as 0, whenever δ is any number between 0 and $\mathbf{u}$. This effectively limits the power of the singularity of the Gauss map.

To return to the analogy of the introduction, we consider the domain of machine representable numbers not as a smooth circle but as a slotted ring, with the number of slots on the ring corresponding to the number of machine-representable numbers in the interval $[0,1)$, where all numbers in $[0,\mathbf{u})$ are "lumped together". In this analogy, $\mathbf{u}$ corresponds to the approximate width of the slots. Now our bead can only occupy one of the slots on the ring, and not just any arbitrary position, and the floating-point Gauss map takes the bead from one slot, winding around the ring as many times as are indicated by the integer part, and finally putting the bead into another slot. We see now that the maximum winding number of the floating-point Gauss map is finite, and the slot next to the origin is the one with this winding number.

A more evident difficulty is that all of the representable points are *rational*, and we know that the exact Gauss map takes these initial points to zero eventually. So if we define a floating-point Gauss map as

$$\hat{G}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} \bmod 1 & \text{otherwise.} \end{cases}$$

where now the operations of division and "mod 1" take place over the floating-point domain, with attendant roundoff error, we have to answer some new dynamical questions:

1. Are there any orbits which *don't* go to 0?
2. Is the behaviour of the floating-point Gauss map similar to the exact Gauss map? In particular, is $\hat{G}$ chaotic?
3. Can we define an appropriate Lyapunov exponent for this map?
4. Is numerical work with $\hat{G}$ useful at all for study of G?

Not surprisingly, some orbits under $\hat{G}$ do terminate at 0, though often not when we expect them to. However, on some machines, some orbits never hit 0, being periodic. For example on the HP28S the initial point $\gamma_0 = 0.3$ gives $\gamma_1 = 0.3333333333$, $\gamma_2 = 0.0000000003$, and $\gamma_3 = 0.3 = \gamma_0$, with period 3. Note that under the exact Gauss map the second iterate (γ_2) of this initial point is zero. Since the set of machine-representable numbers is finite, *all* orbits are ultimately periodic (perhaps with period 1, as at $x = 0$). Note that the behaviour of $\hat{G}$ depends strongly on the floating-point implementation. For example, with the Apple SANE numerics implementation, the starting point $\gamma_0 = 0.3$ gives an orbit with either a long transient regime or a long period, with no regularity detected in the first 65,000 elements of the orbit.

Since all orbits are ultimately periodic, and there are only a finite number of such orbits, the floating-point Gauss map (and indeed *any* machine simulation of *any* dynamical system) is **not** chaotic in the usual sense. Arbitrarily small perturbations in the initial conditions are not even allowed, so the sensitivity of the map to such perturbations is moot. The definition of the Lyapunov exponent for the exact Gauss map seems not to be relevant here: the presence of the derivative $G'(x)$ in the definition of Lyapunov exponent measures the effect of such arbitrarily small perturbations. However, if we define an approximate Lyapunov exponent for the first N iterations of the floating point Gauss map as

$$\lambda_N(\gamma) = \frac{1}{N}\ln\left(\prod_{i=0}^{N}|\hat{G}'(\gamma_i)|\right),$$

whenever the elements of the orbit are nonzero, then this in some sense measures the average sensitivity of the first N elements of the corresponding orbit under the *exact* Gauss map to arbitrarily small perturbations. This "Lyapunov exponent" is what is calculated in practice for a great many numerical simulations of dynamical systems, and if it is positive this is taken as evidence for chaos in the underlying system (Guckenheimer and Holmes [1985]).

But what if the calculated orbit has no counterpart in the exact system? If roundoff errors introduced into the calculation produce an orbit that is unlike any in the exact system, this approximate Lyapunov exponent would be spurious. We will give a proof in the following section, which uses the techniques of backward error analysis, that shows orbits under the floating-point Gauss map are "machine close" to corresponding orbits under the exact Gauss map. A general theorem of this nature has been proved for hyperbolic invariant sets, by Bowen (Guckenheimer

& Holmes [1985]). Here a direct proof is more appropriate and informative. This means that the approximate Lyapunov exponent defined above will accurately reflect the Lyapunov exponent of *some* orbit of the exact Gauss map, provided N is large enough that transient effects have been minimized, and not so large that accumulated roundoff error in the sum degrades the result.

We contrast this behaviour with what happens when continuous maps are made discrete by (e.g.) finite difference schemes. Yamaguti & Ushiki [1981] and Ushiki [1982] have shown that finite difference formulae applied to non-chaotic continuous systems may produce chaotic numerical solutions if the stepsize h is not too small, assuming the calculations are carried out *exactly*. In contrast we have shown here that a chaotic discrete map becomes nonchaotic when implemented in fixed-precision arithmetic.

A further difficulty is that all of the orbits of $\hat{G}$ are ultimately periodic, and periodic orbits of G have Lyapunov exponents that are different from the almost-everywhere value (which is usually the exponent of physical interest). It is not immediately clear that these Lyapunov exponents calculated from $\hat{G}$ will tell us anything useful about the exact map G.

On closer examination, however, we see that if the period of an orbit is long, then the orbit behaves for a long time as if it were aperiodic, reflecting the effect of "nearby" initial points that are aperiodic. Hence we may expect that the computed Lyapunov exponent of a long period orbit will be close to $(\pi^2/6)\ln 2 = 2.373\ldots$. This is what happens in practice, since many initial points seem to give long period orbits. For example, if we compute the first 100,000 elements of the orbit of 0.73 under $\hat{G}$ on the HP28S, we get a computed $\lambda = 2.36992$. This is within 0.2% of the expected value of the Lyapunov exponent of the exact Gauss map. Notice, though, that the Lyapunov exponent of the orbit of the exact map G starting at 0.73 is not even *defined*—we *rely* on the roundoff error to give us our results, which is somewhat unusual. We will expand more on this in a later section.

Orbits under $\hat{G}$ are close to orbits under G. The following theorem justifies the remarks of the previous section. The basic idea of its proof is that given some initial point $\hat{y}$ the floating-point Gauss map also generates an initial point y whose continued fraction is exactly equal to $[a_1, a_2, a_3, \ldots]$, where the a_k are all (machine representable) integers. This initial point y has a G-orbit that is everywhere within a small multiple of $\mathbf{u}$, the machine epsilon, of the $\hat{G}$-orbit of $\hat{y}$. The technique of the proof is of interest for more than just the Gauss map, because similar techniques can be used to prove that numerical simulations of orbits of some continuous systems are machine close to exact orbits of some nearby initial point (for a descriptive review of work by Yorke, Grebogi, and Hammel establishing similar results for continuous maps see Cipra [1988]).

Theorem. *If $x_0, x_1, x_2, x_3, \ldots$ is the sequence of iterates of $\hat{G}$, and $a_1, a_2, a_3, \ldots$ is the sequence of (machine representable) integers that arise in the process, then $y = [a_1, a_2, a_3, \ldots]$ has an orbit under G whose elements are close to $x_0, x_1, x_2, x_3, \ldots$ in a sense to be made precise, and, in particular, y is close to x_0.*

We will show first that we may approximate an element of the orbit of y by a certain rational number. We then show, using a common model of floating-point arithmetic, that the corresponding x_k is "machine close" to this same rational number. This last will be seen to depend on the fact that if you run the Gauss map backwards, errors are damped instead of amplified.

Proof: Consider $y_k = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots]$. The rational numbers $p_n/q_n = [a_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+n}]$ satisfy

$$\left| y_k - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$$

and $q_n \geqslant F_n$ where F_n is the nth Fibonnacci number, from elementary properties of simple continued fractions (see Olds [1963] or Hardy and Wright [1979] for details). This means that given an $\varepsilon > 0$, we can find an n so that $|y_k - p_n/q_n| < \varepsilon$.

To prove the second part, we use the common model of floating-point division that states that if the floating point numbers a, b, and c satisfy $a \div b = c$, where the division takes place over the floating-point numbers, then there is a number δ with $|\delta| < \mathbf{u}$ so that $c(1 + \delta) = a/b$ exactly. Note that we do not model the addition, since this will be seen to be unnecessary.

If the orbit $x_0, x_1, x_2, x_3, \ldots$ has been produced by a floating-point system satisfying this model, then for each n there is a number δ_{k+n} with $|\delta_{k+n}| < \mathbf{u}$ such that

$$(1 + \delta_{k+n}) x_{k+n} = \frac{1}{a_{k+n+1} + x_{k+n+1}},$$

where we may consider the addition as exact, since a_{k+n+1} is a machine representable integer, defined by this process, and x_{k+n+1} is a machine representable floating point number. If we put $\varepsilon_{k+n+1} = x_{k+n+1}/a_{k+n+1}$ then we have

$$(1 + \varepsilon_{k+n+1})(1 + \delta_{k+n}) x_{k+n} = \frac{1}{a_{k+n+1}}.$$

Now put $z_{k+m} = [a_{k+m+1}, a_{k+m+2}, a_{k+m+3}, \ldots, a_{k+n+1}]$ for $m = 1, 2, \ldots, n$, and put $\varepsilon_{k+m} = z_{k+m} - x_{k+m}$ for $m = 0, 1, 2, \ldots, n$. Note that $\varepsilon_k = z_k - x_k$ is the error we wish to estimate, since by the first part we can estimate the error $z_k - y_k$. So now

$$(1 + \delta_{k+m}) x_{k+m} = \frac{1}{a_{k+m+1} + x_{k+m+1}} = \frac{1}{a_{k+m+1} + z_{k+m+1} - \varepsilon_{k+m+1}}$$
$$= z_{k+m} \cdot \frac{1}{1 - \varepsilon_{k+m+1} \cdot z_{k+m}}$$

from whence, on cross-multiplying and expanding, we get the recurrence relation

$$\varepsilon_{k+m} = \delta_{k+m} x_{k+m} - (1 + \delta_{k+m}) z_{k+m} x_{k+m} \varepsilon_{k+m+1}$$

from which we may derive an upper bound on $\varepsilon_k = z_k - x_k$, and we note at this point that z_k is one of the rationals which approximates y_k. Note that the first term in this recurrence relation is essentially the roundoff error introduced at this particular step, while the second term is the error from one level below in the continued fraction, multiplied by a "shrinkage factor" $z_{k+m} x_{k+m}$.

As in the proof that τ has the minimum Lyapunov exponent, we are unable to say anything useful about z_{k+m} directly, but we are able to bound $z_{k+m} z_{k+m+1}$, which is easily shown to be less than 1/2. With some simple estimates on the above recurrence this gives

$$\varepsilon_{k+m} \leqslant \begin{cases} 4\mathbf{u} + \dfrac{1-4\mathbf{u}}{2^{(n+1-m)/2}} & n-m \text{ is odd} \\ 4\mathbf{u} + \dfrac{1-3\mathbf{u}}{2^{(n-m)/2}} & n-m \text{ is even} \end{cases}$$

and since as $n \to \infty$, $z_k \to y_k$, we have at last

$$|x_k - y_k| \leqslant 4\mathbf{u}.$$

Thus there is a nearby initial point y_0 whose orbit under G follows as near as can be expected the computed orbit $x_0, x_1, x_2, x_3, \ldots$ of the floating-point Gauss map.

Our earlier example of $x_0 = 0.3$ gave a periodic orbit on the HP28S, which has $\mathbf{u} = 10^{-11}$. The nearby initial point with this orbit under $\hat{G}$ is

$$\begin{aligned} y &= [3, 3, 3333333333, 3, 3, \ldots] \\ &= \frac{1}{2}(\sqrt{111111111111288888888889} - 33333333333) \\ &= 0.3 + .2999999999976 \cdot 10^{-12} + \cdots. \end{aligned}$$

As a further curiosity, we note that the machine representation of $1/\tau$ on the HP28S is an actual fixed point of $\hat{G}$, allowing us to calculate the exact continued fraction of $1/\tau$ on a finite machine.

A New method for calculating π. The observation that we can get an approximate value for the Lyapunov exponent of the exact Gauss map by calculating the average exponent from the first N elements of a numerically generated orbit gives us a new and interesting, though completely impractical, method for calculating π. We simply choose some initial point more or less at random, say $x_0 = 0.73$, and produce the first N iterates under the floating-point Gauss map, and accumulate the average Lyapunov exponent. At the end, this is supposed to be close to the exact almost-everywhere Lyapunov exponent of the exact Gauss map, $\pi^2/6\ln 2 = 2.373\ldots$. Well, if we know $\ln 2$ and can take square roots, this gives us the value of π. Using the HP28S and 100,000 iterates of the floating-point Gauss map with the above initial point, we get $\pi \approx 3.13945$. Note that this method *relies* on roundoff error, since without it this orbit terminates!

Remarks. This method is likely worse than nearly any other in existence, since it does **not** converge to the correct value in any particular fixed-precision system, since all orbits are ultimately periodic, and the Lyapunov exponent of a periodic orbit is the logarithm of an algebraic number, which can't be π^2 unless e^{π^2} is an algebraic number[2]. Yet this qualifies as a genuine method, since in principle you could implement higher and higher precision floating-point systems and achieve the desired accuracy by longer and longer runs with this high-precision arithmetic. Of course this is impractical, perhaps even ridiculous. There is also the problem of choosing "good" initial points—if we are lucky, the first initial point we choose for whatever floating-point system we have will do the trick—but there is no guarantee, and indeed the computed Lyapunov exponent may converge to something totally different (or worse, something only slightly different).

[2]This is a well known unsolved problem.

This method is clearly related to the Monte Carlo methods, with the roundoff error associated with the floating-point arithmetic playing the part of the random number generator required. The author knows of no other case in mathematics where roundoff error plays a useful role in an actual calculation.

6. CONCLUSIONS. The Gauss map has been shown to be a good example of a chaotic discrete dynamical system, in that it exhibits in an accessible fashion all the common features of such systems. The map is simple enough that the relationship of numerical simulation of the map to the exact map can be explored effectively. We find that the numerical simulation of the map behaves significantly differently, in that the numerical simulation is not chaotic, but is still useful in that the Lyapunov exponent of the exact map can be accurately calculated from the simulation. We have in fact shown that this behaviour of numerical simulation is general. We have also exhibited a new (though impractical) method for the calculation of π.

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> Civilization advances by extending the number of important operations which we can perform without thinking.
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> —*Alfred North Whitehead*